In the previous lecture, it was shown that it is possible to compute the wave drag of wing-body combinations with a reasonable degree of accuracy utilizing the theory for bodies of revolution. In this lecture the theoretical methods for computing the drag of bodies of revolution will be discussed.

Since we wish to design airplanes with minimum wave drag, the theoretical methods for arriving at bodies of revolution with minimum drag for certain fixed conditions will also be considered. Such bodies determined by non-slender theory as well as slender body theory will be discussed.

If it is assumed that the configuration is slender, then the wave drag for a body of revolution at supersonic speeds is defined by the well-known Karman formula for the wave drag of a slender body (ref. ) which may be written as

\[ \frac{D}{q} = -\frac{1}{2n} \int \int S''(x)S''(\xi) \left( n k \xi x \right) - \frac{2}{3} \int d\xi dx \]

where \( x \) and \( \xi \) are two different locations along the \( x \) axis and \( S \) is the cross-sectional area at these stations and \( S''(x) = \frac{d^2S(x)}{dx^2} \)

Following Sears (Ref. ) we may expand \( S(x) \) in a Fourier series and obtain in this way a formula for the drag which is completely analogous to
the well-known formula for the induced drag of a wing in terms of its spanwise load distribution. Thus, if we write

\[ x = \frac{1}{2} \cos \varphi \]

and

\[ S_1(x) = \frac{dS}{dx} = \frac{1}{2} \sum_{n=1}^{N} A_n \sin n\varphi \]

we obtain for the wave resistance

\[ \frac{D}{q} = \sum_{n=1}^{N} n A_n^2 \]

in which

\[ A_n = \frac{2}{\pi} \int_{0}^{\pi} S_1(x) \sin n\varphi \, \text{d}\varphi \]

Of all the terms of the series, each contributes to the drag, but only \( A_1 \) and \( A_2 \) contribute the volume or the base area of the system. Thus, to achieve a small drag with a given area or with a given overall volume within a given length, the higher harmonics in the curve \( S_1(x) \) should be suppressed. This formula enables us to characterize the smoothness of a given shape in a qualitative manner.

Computation of the wave drag by the Fourier series is relatively complex. To obtain an answer in a reasonable length of time requires the use of a high-speed computer. Recently, Cahn and Olstad of the Langley 8-Foot Transonic
Tunnels Branch have developed a numerical method for evaluating the Karman wave-drag equation. This method may be set up easily for a desk calculator and electronic computer and answers are obtained in a much shorter period of time than those obtained by the Fourier series. Comparisons of results obtained by this method indicate that it may provide more satisfactory calculations of the real wave drag than do solutions utilizing the Fourier series. This relatively new method will now be discussed in some detail.
The wave drag equation for a body of revolution as given in reference 3 can be presented in the form

\[ \frac{D}{q} = -\frac{1}{\pi} \int_{0}^{2} \int_{0}^{x} S'(x)S''(\xi) \ln(x - \xi) d\xi dx \]  

(1)

Since this equation holds only when the slope of the cross-sectional area distribution is zero at the front and rear of the body equation (1) may be non-dimensionalized as follows:

\[ \frac{D}{q} = -\frac{L^2}{\pi} \int_{0}^{1} \int_{0}^{x} \frac{S'(x)S''(\xi)}{\xi^2} \ln\left(1 - \frac{\xi}{\xi} \right) \frac{d\xi}{\xi^2} d\xi dx \]  

(2)

The above integral may be considered as the volume between a surface determined by the function \( S'(x)S''(\xi) \ln(x - \xi) \) and the \( x, \xi \)-plane. The volume is bounded by the plane \( \xi = 0, x = L \), and \( x = \xi \) (see sketch).

Along any line \( x - \xi = \text{constant} \) the term \( \ln(x - \xi) \) is constant. Thus, if the first integration proceeds along this line, the term \( \ln(x - \xi) \) may be taken outside of the integral sign. The second integration is then performed with respect to \( x - \xi \) from zero to \( L \).

Considering the above, a numerical solution can be somewhat simplified.

The \( x, \xi \)-plane can be divided into a number of finite squares of equal area.
and a value which is equal to \( S^n(x)S^n(\frac{x}{2}) \) at the center of the square can be assigned to each. These values then are summed along the diagonal \((x - \frac{x}{2} = \text{const.})\) and multiplied by the average value of \( \ln(x - \frac{x}{2}) \). The average value of \( \ln(x - \frac{x}{2}) \) for any set of squares on the line \( x - \frac{x}{2} = \text{constant} \) is obtained by integrating \( \ln(x - \frac{x}{2}) \) over any square of the desired set and dividing by the area of the square. Using the average value of \( \ln(x - \frac{x}{2}) \) rather than the value on the line itself avoids the problem of the singularity on the line \( x = \frac{2}{5} \). It should be noted here that the summation for the line \( x = \frac{2}{5} \) is divided by two so that no areas outside of the limits of the integration are included. Finally, the products of the summation along each \( x - \frac{x}{2} = \text{constant} \) line and the average value of \( \ln(x - \frac{x}{2}) \) are summed to obtain the solution. This solution is described by the following expression:

\[
\frac{D}{q} = -\frac{2}{\pi n^2} \sum_{j=0}^{n} \ln \Delta_j \sum_{i=j}^{n} S_i'' S_{i-j}''
\]

Where \( \ln \Delta_j = (j + \frac{1}{2}) \ln(j + \frac{1}{2}) - (j - \frac{1}{2}) \ln(j - \frac{1}{2}) \) leave out for \( j \) from 1 to \( n \).

When \( j = 0 \) \( \ln \Delta_{j=0} = \frac{1}{2} \ln \frac{1}{2} \)
In order to determine the accuracy of the numerical method the wave
drag of an analytic body shape was computed by this method and by direct
solution of von Karman's equation. The shape of the analytic body was given
by the following expression:

\[ \frac{r}{h} = 4 \left[ \frac{x}{2} - \left( \frac{x}{2} \right)^2 \right] \]

The value of the wave drag coefficient obtained by the numerical method was
42.688(R/2)^2 as compared to the exact value of 42.667(R/2)^2 obtained by
direct integration. A layout of the calculations involved in the numerical
method is presented in Table I. In figure 1, the theoretical value of wave
drag determined from above is plotted on/experimental curve of drag versus
Mach number, from reference 4.

The numerical method was also applied to determine the wave drag coefficient
based on the frontal area of the non-analytic shape of figure 2. The value
obtained was 0.1775. This value is plotted on figure 2 along with experimental
data, from reference 4. Also shown in figure 2 is the drag computed using the
method of reference 2. It can be seen that the method of this report gives
a closer approximation to the actual value of drag. The discrepancy between
the two methods is attributed to the difficulty caused by a discontinuity
in the body area distribution which seems to have a greater effect on the
Fourier series solution.

For both numerical solutions investigated, the x and y axes were each
divided into 40 equal increments. The results for the "bumpy" non-analytic
shape indicate that 40 is a sufficient number of increments necessary to
obtain a good degree of accuracy. In order to facilitate future calculations
a compilation of values of 2n \Delta p for n = 40 is given in Table II. A
larger number of increments may be used for a body which has rapid changes
in shape. However, it should be kept in mind that such a body will not permit
linearized flow approximations and the original equation should not be expected
to yield a solution in good agreement with experiment.

When the slope of the area distribution at the base is different from
zero, additional terms must be used with the von Karman equation (see ref. 5).
These terms are:

\[
\frac{\left[S'(z)\right]^2}{2\pi} \ln \frac{z}{\rho r(z)} + \frac{S'(z)}{\pi} \int_0^l S''(x) \ln (z-x) \, dx
\]  

(4)

The above integral can be solved by a single numerical summation utilizing
the information already obtained in the evaluation of equation (1).

It should be noted that the technique developed here can be readily
adapted to the evaluation of the wave drag of lifting configurations
(see ref. 6) and to the vortex drag of a lifting surface in subsonic or
supersonic flow (see ref. 5).
Let us now consider the problem of determining the minimum wave drag for bodies of revolution with certain conditions fixed. The solution utilizing slender-body theory will be considered first. In the design of an airplane, a number of different conditions may be fixed. For example, length, volume, maximum cross-sectional area, and areas at other points along the length. The position of maximum cross-sectional area may vary along the length.

The solution for the specific fixed conditions of a fixed length and a fixed volume is the simplest solution to be arrived at. For this case, the radii at the various stations along the body are defined by the formula

\[
\frac{r}{r_{\text{max}}} = \left[ 1 - \left( \frac{x}{x_0} \right)^2 \right]^{3/4}
\]

The drag coefficient for this condition is defined by the formula

\[
C_D = \frac{D}{\frac{1}{2} \rho V^2 S_{\text{max}}} = \frac{\rho g}{8} 2 \left( \frac{r_{\text{max}}}{x_0} \right)^2
\]

However, there are relatively few practical cases for which these simple fixed conditions will be used. Normally, the design of an airplane is fixed by certain cross-sectional areas rather than by a fixed volume. The formula for the drag of a closed body for which the maximum fixed cross-sectional
Several people including W. T. Lord and E. Eminton have provided equations which are sufficiently general to define the minimum wave drag for all usual fixed conditions. The general formula is very complex and beyond the scope of the present discussion; however, it might be of interest to consider some of the results obtained from this analysis. Presented in figure 3 are the optimum area distributions for some special cases in which the nose is pointed and the base has a finite area. Curve 1 is for a fixed base area only. The shape is the same as that of the projectile noses developed by von Karman and others. Curve 2 is for a fixed volume equal to the base area times the length. Curve 3 is for a fixed area equal to the base area at a distance 0.3 of the length from the nose. Curve 4 is for the same fixed area and for the same volume.
of curve 3. The most interesting point to note is that the minimum drag is obtained in some cases by increasing the maximum area above that for the maximum fixed area. For example, as shown in curve 3. For most practical cases, the point of maximum fixed area is not so extremely located and the position for the maximum re-located total area approaches that for the fixed area.

The previous discussion of the minimum drag bodies has been based on slender-body theory. For the Mach numbers and fineness ratios for which the area rule seems to offer the most promise for reducing the drag, the body cannot be considered slender. To arrive at the shape that provides the minimum wave drag non-slender theory must be utilized.

Ferrari (refs. 4 and 5) has considered the minimum-drag problem for the length-caliber body and the ducted body on the basis of linear theory without resorting to the slender-body approximation. Because he used
the linearized drag integral in the triple-integral form, Ferrari found it necessary to use a series expansion for the source-distribution function, which introduced considerable complexity at an early stage in the analysis. Recently Parker (ref. 6) reduced the linearized drag integral for bodies of revolution to a double-integral form and determined, without resort to slender-body theory, the minimum-wave-drag shape for a transition section between two semi-infinite cylinders, a special case of which is the projectile tip. Parker has recently formally solved the problem of the three-point body of revolution that has minimum wave drag, based on linearized supersonic-flow theory, without resort to slender-body approximations. The source-distribution function involves an integral which apparently cannot be evaluated in terms of simple functions and for which a series expansion may be used. Numerical quadratures are required to determine the shape of the body.

Complete calculations have been made for a closed body with a given section $\beta R_a = 0.2$ at the midbody position $a = 0.5$. The shape of the body is given in Figure 3. An inspection of the mathematics shows that the corner is present for any finite value of $R_a$. In the slender-body limit ($\beta R_a$ approaching zero) the corner vanishes as $2\pi (\beta R_a)^2$. 
Complete calculations have been made for a closed body with a given section $R_a = 0.2$ at the midbody position $a = 0.5$. Figure 4 permits an easy comparison of the minimum-drag body thus computed with other minimum-drag bodies which have been computed using slender body theory. The ordinate is $(R_{\text{body}} - R_{\text{cone}})/R_{\text{max}}$ where $R_{\text{body}}$ is the radius of the body, $R_{\text{cone}}$ is the radius of the cone whose vertex is at the nose of the body and whose base is the maximum cross section of the body, and $R_{\text{max}}$ is the radius at the maximum section. The abscissa $x$ is distance along the body and the maximum section is at $x = 0.5$. The figure shows (A) the forward and (E) the rearward portions of the body computed herein, (C) the corresponding linear-theory minimum-drag projectile tip, (D) the Sears-Haack symmetrical length-caliber body, and (E) the Von Karman projectile tip. The radii of the minimum-drag bodies according to slender-body theory (D and E) are functions of $x$ only, whereas the radii of the minimum-drag bodies according to linear theory (A, B, and C) are functions of $R_a(R_{\text{max}})$ as well as $x$.

Recent unpublished work by Clinton E. Brown of the Langley Aeronautical Laboratory shows that to the order of accuracy of linear theory the wave drag of bodies of revolution is reversible. With the plausible assumption that the minimum-drag problem has a unique solution, the body computed here
should be symmetrical. The slight asymmetry found here is not to be interpreted as a refutation of the reversibility theorem, since higher order terms, of appreciable influence because the body chosen is rather thick, undoubtedly are handled (included or omitted) in different ways in the different treatments.
length-caliber body, and (E) the Von Karman projectile tip. The radii of 
the minimum-drag bodies according to slender-body theory (D and E) are 
functions of \( x \) only, whereas the radii of the minimum-drag bodies according 
to linear theory (A, B, and C) are functions of \( \beta Ra(z/R_{max}) \) as well as \( \alpha \).

It is apparent that the lack of symmetry of the body computed herein 
(the difference between curves A and B) is of the same order as the dif-
ference between that body and two linear-theory minimum-drag projectile 
tips placed base to base. Both forward and rearward portions are thicker 
than the projectile tip and their average is thicker than the projectile 
tip in approximately the same proportion that the Sears-Haack length-
caliber body is thicker than the Von Karman projectile tip. In the 
extreme case where \( \beta Ra = 0.5 \) (of course, completely outside the range 
of validity of linear theory) the body computed in this report would become 
two Mach cones and the curves A, B, and C would collapse into the abscissa. 
In the limiting case, \( \beta Ra \to 0 \), the three-point body becomes the Sears-Haack 
length-caliber body; that is, curves (A) and (E) coalesce into curve (D) 
while curve (C) approaches curve (E).

Therefore, in the range of validity of linear theory, the minimum-
drag length-caliber body with fixed caliber in the symmetrical position is thicker in both forward and rearward parts than the corresponding minimum-drag projectile tip. Thus the conclusion is reached that the influence between the forward and rearward portions of a closed body results in a small increase in the radii of the minimum-drag shape.