Dr. Richard Whitcomb  
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Dear Dr. Whitcomb:  

I enclose three copies of a paper Korn and I are  
submitting to the Communications on Pure and Applied  
Mathematics for publication. I hope you can circulate  
them to the interested parties at Langley, notably to  
Jerry South and Ed Garrick.  

I think it would be a good idea if you could come  
up before too long, say in July, to review what we have  
been doing and to discuss our plans for next year. I  
will be away on a short vacation at the end of this month,  
but will get in touch with you by telephone about such a  
visit after I return.  

I hope there is good progress on the experimental  
model of the airfoil we brought you in March.  

Sincerely yours,  

Paul R. Garabedian  

PRG: fbb  
Enclosures
ANALYSIS OF TRANSONIC AIRFOILS

P. R. Garabedian and D. G. Korn*

1. Introduction

Interest has revived recently in transonic aerodynamics because of the advent of airplanes designed to operate just below the speed of sound. In earlier work [2,7] we have developed a hodograph method based on complex characteristics that enables us to calculate supercritical wing sections that will be free of shocks at a specified speed and angle of attack. The purpose of the present paper is to describe a finite difference scheme for the analysis of such transonic airfoils at off-design conditions. Our ultimate goal is to avoid expensive wind tunnel tests by combining these mathematical techniques in a procedure for designing supercritical airfoils so they will be effective over a wider range of angles of attack and Mach numbers.

What we require is a mathematical method for computing two-dimensional transonic flows past a prescribed profile that will be on the one hand capable of yielding accurate results to compare with a known shockless regime, but will on the other hand give quickly data of engineering reliability concerning the location and strength of

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shocks at off-design conditions. Experience shows that the supersonic zone on a shockless airfoil is very sensitive to changes in the angle of attack $\alpha_0$ or the Mach number $M$ at infinity, and that several weak shocks may tend to appear simultaneously. These considerations call for a mathematical formulation of the problem that is based on the theory of weak solutions and artificial viscosity rather than on sharp shock fitting. Moreover, we are not concerned with the accurate description of a single strong shock, since that would in any case cause boundary layer separation, drag rise and buffeting. Therefore we have chosen to ignore variations in the entropy across shocks, and we have even found that it is preferable to dispense with the conservation form of the equations of motion.

The most successful finite difference scheme for handling the difficulties associated with transonic flow seems to be the one advanced by Murman and his collaborators [8,10,11]. We have therefore developed a second order accurate version of that method which is suitable for our purpose. We implement it in a coordinate system shown by Sells [12] to be very effective for subsonic airfoil computations. This employs the familiar step of mapping the interior of the unit circle conformally onto the exterior of the profile, and it leads to a desirable distribution of mesh points on the body as well as to an adequate representation of the flow at infinity. To compute the modulus of the derivative of the map
function, which is really all that is needed, we use a special finite difference scheme that fits in well with the rest of the procedure. Thus we arrive at a simple, rapid program for calculating the velocity potential of the flow as a function of a natural set of coordinates on a rectangular grid of mesh points.

In Murman's scheme shock waves emerge as jumps in the pressure coefficient spread over only one or two mesh intervals. In the case of stronger shock waves we have found that the method converges remarkably well, although for several weaker shocks occurring close together, for example near a shockless design condition, the convergence may be slower and more erratic. In our formulation we have adjusted the truncation error so that it becomes a term of second order in the mesh size that can be interpreted as an artificial viscosity. Both this term and the direction in which the iterative procedure is oriented impose an entropy inequality on the solution obtained. Thus the computational results not only furnish a reliable numerical analysis of the transonic flow under investigation, but also yield a verification of the theorem that a weak solution of the problem should exist and should be unique subject to an appropriate entropy inequality.

To debug our analysis program we have applied it to a shockless profile we designed earlier using the method of complex characteristics [2]. In this work we have been significantly aided by test data on our profile made in the
two-dimensional high Reynolds number wind tunnel of the National Aeronautical Establishment at Ottawa by Jerzy Kacprzynski and Lars Ohman[6]. Comparisons of the analysis with both our original design calculation and the Canadian experiments are presented in the figures at the end of the paper. The remarkably satisfactory agreement gives us every reason to expect not only that shockless airfoils will be a practical guide in the design of supercritical airplane wings, but also that their two-dimensional analysis by mathematical methods may become as acceptable as wind tunnel testing.

2. Formulation of the problem

We are concerned with the numerical solution of the second order partial differential equation

\[(c^2-u^2)\phi_{xx} - 2uv \phi_{xy} + (c^2-v^2)\phi_{yy} = 0\]

for the velocity potential \(\phi\) of a steady compressible plane flow. Here \(u = \phi_x\) and \(v = \phi_y\) are the velocity components, and the speed of sound \(c\) is defined by Bernoulli's law

\[\frac{u^2+v^2}{2} + \frac{c^2}{\gamma-1} = \frac{1}{2} + \frac{M^{-2}}{\gamma-1},\]

where \(\gamma = 1.4\) is the ratio of specific heats. In transonic aerodynamics it is preferable to study the Neumann problem for \(\phi\) instead of a Dirichlet problem for the stream function \(\psi\) because the density \(\rho\) is a multiple-valued function of the
gradient of $\psi$ that branches at the sonic line. We want to calculate the flow past a prescribed profile $C$ on which the boundary condition

$$\frac{\partial \phi}{\partial \nu} = 0$$

is imposed, where $\nu$ stands for the inner normal. At infinity the flow is supposed to have a given subsonic Mach number $M < 1$, with $u = 1$ and $v = 0$, and it has a circulation $\Gamma$ adjusted to satisfy the Kutta-Joukowski condition, which asserts that the speed at the sharp trailing edge of $C$ is finite. We are interested in cases where $M$ and $\Gamma$ are so large that supersonic zones appear on the upper surface of $C$ and perhaps on the lower surface, too. We are looking for a method of solution capable of describing weak shock waves behind these supersonic zones. We assume that changes in entropy across the shocks can be neglected, so we use approximate shock conditions stating that $\phi$ and $\psi$ remain continuous, which are second order accurate in the shock strength.

In this paper we shall not attempt to use the conservation form

$$\left(\rho \phi_x\right)_x + \left(\rho \phi_y\right)_y = 0$$

of the differential equation for $\phi$ because the formula for $\rho$ as a function of $u$ and $v$ obtained from Bernoulli's law
introduces unnecessary complications when we come to specifying an analogous finite difference equation. Moreover, the mean value theorem shows that a central difference approximation to the quasi-linear statement (1) of the equation for \( \phi \) that we do use agrees with the conservation form up to second order terms in the shock strength, so that nothing would be gained by using (4) in our case of weak shocks anyway. However, we shall have to incorporate an appropriate entropy inequality into our numerical solution of the problem in order to be certain of getting a unique answer. This will be achieved by following Murman's implicit finite difference scheme in the supersonic region, which uses expressions for two of the second derivatives of \( \phi \) that are retarded in the direction of motion so as to introduce an artificial viscosity with the right sign.

In order to formulate the Neumann boundary condition on \( \phi \) conveniently and to represent the singularity of the flow at infinity accurately, we map the exterior of the profile \( C \) conformally onto the interior of the unit circle, with the point at infinity corresponding to the origin, and we express the equation for \( \phi \) in terms of polar coordinates \( r \) and \( \theta \) inside the circle. Let

\[
(5) \quad x + iy = P(re^{i\theta}) = \sum_{n=-1}^{\infty} a_n r^n e^{i\theta}
\]

denote the map function, and put
(6) \[ f = r^2 |F'(re^{i\theta})| \quad . \]

Since \( \phi \) still satisfies the conformally invariant conservation law

\[ (\rho \phi_{\theta})_{\theta} + r(x\rho \phi_{\theta})_{\theta} = 0 \]

in the new coordinate system, we find that

\[ (c^2 - r^2 f^{-2}) \phi_{\theta \theta} + 2r f^{-2} \phi_{\theta} \phi_{\theta \theta} + r^2 (c^2 - r^2 f^{-2}) \phi_{\theta} = 0 \]

\[ + r(c^2 - r^2 f^{-2}) \phi_{\theta} + 2r f^{-2} \phi_{\theta} + r^2 (c^2 - r^2 f^{-2}) \frac{\partial^2 \phi}{\partial r^2} = 0. \]

This is the partial differential equation for \( \phi \) that we shall solve over a range \( 0 < r_0 < r < 1 \) including all supersonic points. Over the rest of the flow \( 0 < r < r_0 < 1 \) we substitute

(8) \[ \phi = \phi - \frac{f_0 \cos(\theta + \alpha)}{r}, \]

where \( \alpha = \alpha_0 + \alpha_1 \), \( \alpha_0 \) is the angle of attack, and \( \alpha_1 \) is the argument of the coefficient \( a_{-1} = f_0 e^{-i\alpha_1} \) and is fixed by requiring that the sharp trailing edge of the profile \( C \) correspond to \( \theta = 0 \). Thus \( \phi \) has the boundary values
(9) \[ \phi = \frac{\Gamma}{2\pi} \tan^{-1}\left[\sqrt{1-M^2}\tan(\theta+\alpha)\right] \]

at \( r = 0 \) and fulfills the equation

(10) \[
(c^2-r^2f-\dot{\phi}_\theta^2)\phi_{\theta\theta} - 2r^4f-2\phi_\theta \dot{\phi}_r \phi_r + r^2(c^2-r^4f-\dot{\phi}_r^2)\phi_{rr} - 2r^3f-2\phi_\theta \dot{\phi}_r \phi_\theta
\]

\[
+ r(c^2+r^2f-\dot{\phi}_\theta^2 - 2r^4f-2\dot{\phi}_r^2)\phi_r + f^{-3}(r^2\phi_\theta^2 + r^4\phi_r^2)(f_\theta \dot{\phi}_\theta + r^2f_\phi \dot{\phi}_r)
\]

\[
= f_\phi^3 f^{-3}(r^2\phi_\theta^2 + r^4\phi_r^2) \frac{f_\theta \sin(\theta+\alpha) + rf_r \cos(\theta+\alpha)}{r},
\]

from which the singularities of the flow at infinity have been removed.

In the next section we shall indicate how the partial differential equations for \( \phi \) and \( \dot{\phi} \) can be solved by successive overrelaxation combined with Murman's finite difference scheme. We shall also give an iterative procedure to determine the circulation \( \Gamma \) from the Kutta-Joukowski condition. Section 4 will be devoted to a brief description of our method for computing the modulus \( f \) of the derivative of the map function \( F \). In the final section of the paper we discuss some computations we have made of transonic flows past the profile we designed for testing at the National Aeronautical Establishment in Canada.
3. Transonic difference scheme

In this section we formulate a finite difference scheme for the numerical solution of the partial differential equations (7) and (10). Following Murman, we always march from the nose to the tail of the airfoil. In our coordinate system this means that we advance from \( \theta = \pi \) over the upper surface toward \( \theta = 2\pi \) and from \( \theta = \pi \) over the lower surface toward \( \theta = 0 \). On the inner range \( 0 < r < r_o \) we update each mesh point separately by successive overrelaxation, whereas on the outer range \( r_o < r < 1 \) we use relaxation by columns at each fixed \( \theta \). The relaxation factor is made to vary continuously from a prescribed value just below 2 at \( r = 0 \) to the value 1 in the supersonic zone. The circulation \( \Gamma \) is altered at every cycle by adding to \( \phi \) a multiple of the boundary value term (9) determining by comparing the values of \( \phi \) at \( \theta = 0 \) and \( \theta = 2\pi \) and making them fit the Kutta-Joukowski requirement

\[
(11) \quad \phi_\theta = 0
\]

at the tail. The current value of \( \Gamma \) is used to specify the period of \( \phi \) across the rest of the columns \( \theta = 0 \) and \( \theta = 2\pi \), and data are matched in an obvious way at the interface \( r = r_o \).

At every mesh point \( \theta = j\Delta \theta, \ r = k\Delta r \) we use the centered finite difference approximations
(12) $\phi_\theta = \frac{\phi_{j+1,k} - \phi_{j-1,k}}{2\Delta \theta}$, $\phi_r = \frac{\phi_{j,k+1} - \phi_{j,k-1}}{2\Delta r}$

to the first derivatives of $\phi$, and similarly we put

(13) $\phi_{rr} = \frac{\phi_{j,k+1} - 2\phi_{j,k} + \phi_{j,k-1}}{(\Delta r)^2}$

At all elliptic points, that is, points where the local Mach number is less than one, we use the usual central formulas

(14) $\phi_{\theta\theta} = \frac{\phi_{j+1,k} - 2\phi_{j,k} + \phi_{j-1,k}}{(\Delta \theta)^2}$

and

(15) $\phi_{\theta r} = \frac{\phi_{j+1,k+1} - \phi_{j-1,k+1} + \phi_{j-1,k-1} - \phi_{j+1,k-1}}{4\Delta \theta \Delta r}$

to get finite difference analogues of (7) and, with corresponding rules for $\phi$, of (10). However, to obtain a stable implicit scheme like Murman's in the hyperbolic region $u^2 + v^2 > c^2$, we set

(16) $(\Delta \theta)^2 \phi_{\theta\theta} = (\phi_{j,k} - 2\phi_{j-1,k} + \phi_{j-2,k}) + \epsilon (\phi_{j,k} - 3\phi_{j-1,k} + 3\phi_{j-2,k} - \phi_{j-3,k})$
and

\[ 4\Delta \theta \Delta r \phi_{\theta r} = 2(\phi_{j,k+1} - \phi_{j-1,k+1} + \phi_{j-1,k-1} - \phi_{j,k-1}) + \varepsilon(\phi_{j,k+1} - 2\phi_{j-1,k+1} + \phi_{j-2,k+1} - \phi_{j-2,k-1} + 2\phi_{j-1,k-1} - \phi_{j,k-1}) \]

there for \( \theta > \pi \), with obvious changes when \( \theta < \pi \). If \( \varepsilon = 0 \) this gives an analogue of Murman's scheme that is second order accurate in \( \Delta r \), but only first order accurate in \( \Delta \theta \). By taking

\[ 0 < \varepsilon = 1 - \lambda \Delta \theta < 1 \]

we arrive at a difference approximation that is second order accurate in \( \Delta \theta \), too, and will still be stable for a sufficiently large choice of the parameter \( \lambda \) because the truncation error will be dominated on the right in (7) by the favorable artificial viscosity term

\[ E = \lambda (\Delta \theta)^2 [(c^2 - r^2 \phi_{\theta \theta}) \phi_{\theta \theta} - r^4 \phi_{\phi} \phi_{\phi} \phi_{\theta \theta}] \]

when \( \theta > \pi \), with the sign reversed when \( \theta < \pi \).

We use reflection to handle the boundary condition at \( r = 1 \) by substituting

\[ \phi_{j,k+1} = \phi_{j,k-1} \quad k \Delta r = 1 \]
To find $\phi_{j,k}$ an iterative procedure is set up marching in the direction of the flow. For each fixed $j$ we obtain a tridiagonal linear system of equations for the unknowns $\phi_{j,k}$ by freezing the expressions for the first derivatives $\phi_\theta$ and $\phi_r$, together with $c$, at their values from the previous cycle. The standard von Neumann test based on Fourier analysis indicates that the iterative scheme is stable, and our choice of the artificial viscosity assures that the right entropy inequality is imposed on the solution. At each cycle we compute a new value of the circulation from the jump relation

$$
\Gamma = \phi_{j-1,k} - \phi_{1,k}, \quad jA\theta = 2\pi, \quad kA\tau = 1,
$$

and we correct $\phi$ accordingly by adding to it everywhere a suitable positive multiple of the resulting increment in $\Gamma$ times the vortex flow (9). This is seen to yield a second order accurate evaluation of the Kutta-Joukowski condition (11). Thus the overall plan is that we advance through the supersonic zone solving implicit equations like those occurring in a mixed initial and boundary value problem, then we proceed through the shock relocating it automatically by means of an artificial viscosity, after which we update the subsonic flow and the circulation by overrelaxation, and then we return to the supersonic region, repeating the whole process until it reaches a desired level of convergence.
4. Conformal Mapping

Before we can implement our transonic flow computation in the \((\theta, r)\)-plane, it is necessary to find the conformal mapping \(F\) of the unit circle onto the exterior of the profile \(C\). For this purpose we have developed a finite difference method that determines the modulus \(f\) of the derivative of the map function throughout the grid where it is needed, using input data which are conveniently and accurately specified by our design program [1, 2]. Let \(K\) denote the curvature of \(C\), defined as a function of the arc length \(s\) measured clockwise from the tail, and let \(\chi\) be the angle of the tangent to \(C\). Then

\[
\frac{\partial}{\partial \theta} F(re^{i\theta}) = ire^{i\theta} F'(re^{i\theta}) = \frac{fe^{\frac{\partial \chi}{r}}}{r},
\]

which shows that

\[
U = \log f, \quad V = \chi + \theta
\]

are conjugate harmonic functions. The Cauchy-Riemann equations assert that

\[
\frac{\partial U}{\partial r} = \frac{\partial V}{\partial \theta} = \frac{d\chi}{ds} + 1
\]

at \(r = 1\). Since \(ds/d\theta = f\) and \(d\chi/ds = K\), it follows that \(U\) satisfies the nonlinear mixed boundary condition...
which provides the key to our mapping procedure.

Because all our airfoils $C$ have sharp trailing edges corresponding to $\theta = 0$, we make the substitution

$$W = \log \left[ \frac{f}{1-re^{i\theta}} \right] = U - \frac{1}{2} \log(1-2r \cos \theta + r^2) .$$

Motivated by Sells' treatment of singularities at $r = 0$, we express Laplace's equation for $W$ in the form

$$W_{\theta\theta} + r^2 W_{rr} + r W_r = 0 ,$$

which lends itself to a difference approximation of the kind introduced in the previous section, with uniform mesh sizes in $r$ and $\theta$. The boundary condition on $W$ at $r = 1$ becomes

$$\frac{\partial W}{\partial r} -(2\kappa \sin \frac{\theta}{2}) e^W = \frac{1}{2}$$

and is easily handled by reflection despite its nonlinearity. At $r = 0$ we impose the convenient normalization $W = 0$, which necessitates rescaling the curvature $\kappa$ so that

$$\int_0^{2\pi} \kappa e^W \sin \frac{\theta}{2} d\theta = -\frac{\pi}{2} .$$
We apply successive overrelaxation to calculate \( W \) on a uniformly spaced rectangular grid of mesh points in the \((\theta, r)\)-plane. At each cycle the representation for the curvature \( \kappa \) as a function of the coordinate \( \theta \) must be updated by comparing the known relationship between \( \kappa \) and the arc length \( s \) on \( C \) with a current numerical evaluation of the formula

\[
s = \int f d\theta = 2 \int e^W \sin \frac{\theta}{2} d\theta.
\]

This is an iterative procedure more usually formulated in terms of the conjugation operator specifying \( U \) in terms of \( V \) on the boundary of the unit circle \([3, 4, 5]\). We have recast it in the manner described because on the one hand values of \( f \) over the interior of the circle are required for our transonic flow computations, and on the other hand in practice \( C \) may have limited differentiability properties, so a method is needed that does not hypothesize too much smoothness on \( \kappa \).

5. **Comparison with design and experiment**

The method we have proposed for transonic flow calculations has been programmed for the C.D.C. 6600 computer at N.Y.U. The pressure distribution on an airfoil can be determined with accuracy adequate for engineering applications in as little as five or ten minutes of machine time for each choice.
of the free stream Mach number $M$ and the angle of attack $\alpha$. Moreover, the results of much finer runs have been compared with our earlier design computations to check that both programs are thoroughly debugged and agree to a prerequisite number of significant digits. The many test runs we made show that the subsonic part of the flow varies only normally with $M$ and $\alpha$, whereas in the supersonic zone the sensitivity of the pressure to changes in $M$ and especially to changes in $\alpha$ is very great. This is consistent with the predictions of existence and uniqueness theorems and conjectures for transonic flow problems [2,9].

In Figure 1 we display a comparison of design, analysis and experimental evaluations of the pressure distribution on the airfoil we studied with Kacprzynski and Ohman [6]. In Figure 2 we present a similar comparison of experiment and analysis at a nearby off-design condition where two weak shocks appear. The agreement with the wind tunnel test measurements is especially gratifying, although the data show that a wall effect and boundary layer correction must be made by adding approximately .015 to the Mach number $M$ and .9 degrees to the angle of attack $\alpha$ to simulate the design condition. In the analysis computations mesh sizes $\Delta r = 1/30$ and $\Delta \theta = 2\pi/160$ were used with an artificial viscosity $\lambda = 4$, and each run took about five minutes of machine time. The agreement we obtain using conformal mapping and the exact equations of motion seems to be better than that shown by
Krupp and Murman [8], who used linearized boundary conditions and differential equations because of their interest in going on to do three-dimensional analysis. The wave drag coefficient $C_D$ computed in our analysis program by numerical integration of the pressure coefficient $C_p$ turned out to be $C_D = 0.002$ in the design case of Figure 1, but $C_D = 0.008$ in the off-design calculation of Figure 2. Finally, it seems that boundary layer effects are not so large as to make our inviscid computations unrealistic. However, we hope to return to more detailed study of these questions, including a boundary layer correction and three-dimensional work, in later publications (cf. [1]).
References


Theory: $M = 0.75$, $C_L = 0.63$, $T/C = 0.12$

Experiment: $M = 0.765$, $C_L = 0.576$, $ALP = 89^\circ$, $R = 21 \times 10^6$

Figure 1. SHOCKLESS FLOW
THEORY:  \( M = 0.76, \ C_L = 0.54, \ \text{ALP} = -0.45^\circ, \ T/C = 0.12 \)

EXPERIMENT:  \( M = 0.772, \ C_L = 0.48, \ \text{ALP} = 0.44^\circ, \ R = 20 \times 10^6 \)

Figure 2. OFF-DESIGN CONDITION