In the previous lecture, it was shown that it is possible to compute the wave drag of wing-body combinations with a reasonable degree of accuracy utilizing the theory for bodies of revolution. In this lecture, the theoretical methods for computing the drag of bodies of revolution will be discussed. Since we wish to design airplanes with minimum wave drag, the theoretical methods for arriving at bodies of revolution with minimum drag for certain fixed conditions will also be considered. Such bodies determined by non-slender theory as well as slender body theory will be discussed.

If it is assumed that the configuration is slender, then the wave drag for a body of revolution at supersonic speeds is defined by the well-known Karman formula for the wave drag of a slender body (ref. 1) which may be written as

\[ \frac{D}{q} = -\frac{1}{2\pi^{2}} \int_{0}^{\ell} S''(x)S''(\xi) \ln \frac{|x-\xi|}{\ell} d\xi dx \]

where \( x \) and \( \xi \) are two different locations along the \( x \) axis and \( S \) is the cross-sectional area at these stations and \( S''(x) = \frac{d^{2}S(x)}{dx^{2}} \).

Following Sears (Ref. 2) we may expand \( S'(x) \) in a Fourier series and obtain in this way a formula for the drag which is completely analogous to
the well-known formula for the induced drag of a wing in terms of its spanwise load distribution. Thus, if we write

\[ x = \frac{1}{2} \cos \phi \]

and

\[ s^1(x) = \frac{dS}{dx} = \frac{1}{2} \sum_{n=1}^{N} A_n \sin n\phi \]

we obtain for the wave resistance

\[ \frac{D}{q} = \sum_{n=1}^{N} n A_n^2 \]

in which

\[ A_n = \frac{2}{n} \int_{0}^{\pi} s^1(x) \sin n\phi \, d\phi \]

Of all the terms of the series, each contributes to the drag, but only \( A_1 \) and \( A_2 \) contribute the volume or the base area of the system. Thus, to achieve a small drag with a given area or with a given overall volume within a given length, the higher harmonies in the curve \( s^1(x) \) should be suppressed. This formula enables us to characterize the smoothness of a given shape in a qualitative manner.

Computation of the wave drag by the Fourier series is relatively complex. To obtain an answer in a reasonable length of time requires the use of a high-speed computer. Recently, Cahn and Olstad of the Langley 8-Foot Transonic
Tunnels Branch have developed a numerical method for evaluating the Karman wave-drag equation. This method may be set up easily for a desk calculator and electronic computer and answers are obtained in a much shorter period of time than those obtained by the Fourier series. Comparisons of results obtained by this method indicate that it may provide more satisfactory calculations of the real wave drag than do solutions utilizing the Fourier series. This relatively new method will now be discussed in some detail.
The wave drag equation for a body of revolution as given in reference 3 can be presented in the form

\[ \frac{D}{q} = \frac{-1}{\pi} \int_{0}^{2} \int_{0}^{x} S''(x)S''(\xi) \ln(x - \xi) d\xi dx \]  

(1)

Since this equation holds only when the slope of the cross-sectional area distribution is zero at the front and rear of the body equation (1) may be non-dimensionalized as follows:

\[ \frac{D}{q} = \frac{2^2}{\pi} \int_{0}^{1} \int_{0}^{x} \frac{S''(\xi)}{S''(\xi)} \ln(\frac{x}{2} - \xi) d\xi d(\frac{x}{2}) \]  

(2)

The above integral may be considered as the volume between a surface determined by the function \( S''(x)S''(\xi)\ln(x - \xi) \) and the \( x, \xi \)-plane. The volume is bounded by the plane \( \xi = 0, x = l \), and \( x = \xi \) (see sketch).

Along any line \( x - \xi = \) constant the term \( \ln(x - \xi) \) is constant. Thus, if the first integration proceeds along this line, the term \( \ln(x - \xi) \) may be taken outside of the integral sign. The second integration is then performed with respect to \( x - \xi \) from zero to \( l \).

Considering the above, a numerical solution can be somewhat simplified. The \( x, \xi \)-plane can be divided into a number of finite squares of equal area.
and a value which is equal to \( S''(x)S''(\xi) \) at the center of the square can be assigned to each. These values then are summed along the diagonal \((x - \xi = \text{const.})\) and multiplied by the average value of \( \ln(x - \xi) \). The average value of \( \ln(x - \xi) \) for any set of squares on the line \( x - \xi = \text{constant} \) is obtained by integrating \( \ln(x - \xi) \) over any square of the desired set and dividing by the area of the square. Using the average value of \( \ln(x - \xi) \) rather than the value on the line itself avoids the problem of the singularity on the line \( x = \xi \). It should be noted here that the summation for the line \( x = \xi \) is divided by two so that no areas outside of the limits of the integration are included. Finally, the products of the summation along each \( x - \xi = \text{constant} \) line and the average value of \( \ln(x - \xi) \) are summed to obtain the solution. This solution is described by the following expression:

\[
\frac{D}{q} = -\frac{2}{\pi n^2} \sum_{j=0}^{n} \ln \Delta_j \sum_{i=j}^{n} S_i'' S_{i-j}''
\]

where \( \ln \Delta_j = (j + \frac{1}{2}) \ln(j + \frac{1}{2}) - (j - \frac{1}{2}) \ln(j - \frac{1}{2}) \) for \( j \) from 1 to \( n \).

When \( j = 0 \), \( \ln \Delta_{j=0} = \frac{1}{2} \ln \frac{1}{2} \).
In order to determine the accuracy of the numerical method the wave
drag of an analytic body shape was computed by this method and by direct
solution of von Karman's equation. The shape of the analytic body was given
by the following expression:

\[ \frac{r}{R} = h \left[ \frac{x}{L} - \left( \frac{x}{L} \right)^2 \right] \]

The value of the wave drag coefficient obtained by the numerical method was
42.68(R/L)^2 as compared to the exact value of 42.667(R/L)^2 obtained by
direct integration. A layout of the calculations involved in the numerical
method is presented in Table I. In figure 1, the theoretical value of wave an
drag determined from above is plotted on experimental curve of drag versus
Mach number, from reference 1.

The numerical method was also applied to determine the wave drag coefficient
based on the frontal area of the non-analytic shape of figure 2. The value
obtained was 0.1775. This value is plotted on figure 2 along with experimental
data, from reference 1. Also shown in figure 2 is the drag computed using the
method of reference 2. It can be seen that the method of this report gives
a closer approximation to the actual value of drag. The discrepancy between
the two methods is attributed to the difficulty caused by a discontinuity
in the body area distribution which seems to have a greater effect on the
Fourier series solution.

For both numerical solutions investigated, the x and \( \xi \) axes were each
divided into 40 equal increments. The results for the "bumpy" non-analytic
shape indicate that 40 is a sufficient number of increments necessary to
obtain a good degree of accuracy. In order to facilitate future calculations
a compilation of values of \( Z_n A \) for \( n = 40 \) is given in Table II. A

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larger number of increments may be used for a body which has rapid changes in shape. However, it should be kept in mind that such a body will not permit linearized flow approximations and the original equation should not be expected to yield a solution in good agreement with experiment.

When the slope of the area distribution at the base is different from zero, additional terms must be used with the von Karman equation (see ref. 5). These terms are:

$$\frac{[S'(2)]^2}{2\pi} \ln \frac{\rho}{r(2)} + \frac{S'(2)}{\pi} \int_0^2 S''(x) \ln (2-x) dx$$

(4)

The above integral can be solved by a single numerical summation utilizing the information already obtained in the evaluation of equation (1).

It should be noted that the technique developed here can be readily adapted to the evaluation of the wave drag of lifting configurations (see ref. 6) and to the vortex drag of a lifting surface in subsonic or supersonic flow (see ref. 5).